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RESEARCH ARTICLE

A class of geometric quadratic stochastic operator on countable state space and its regularity

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Abstract

We have constructed a Geometric quadratic stochastic operator generated by 2-partition ξ of singleton defined on countable state space X, where $X=\{0,1,2,...\}$. We have studied the trajectory behavior of such operator for any initial measure $\mu\in S(X,F)$. It is shown that such operator converges to a fixed point which indicates the existence of the strong limit of the sequence $V^n(\mu)$. This follows that such operator is a regular transformation.

Keywords: Singleton, countable state space, quadratic stochastic operator, regular transformation

INTRODUCTION

Bernstein's work on population genetics has triggered the study of quadratic stochastic operator since the early of 20th century. The study of quadratic stochastic operator is still advancing even though it is considered as the simplest nonlinear operator. This theory frequently arises in many models in various fields such as biology, physics, game theory, finance, mathematics, and economics.

The quadratic stochastic operator is a mapping of the simplex

$$S^{m-1}=\left\{x=(x_1,x_2,...,x_m)\in\mathbb{R}^m \middle| x_i\geq 0,\sum_{i=1}^{m}x_i=1\right\}, \qquad (1)$$

into itself, of the form

$$V: x_k^{(1)}=\sum_{i,j=1}^{m}P_{ij,k}x_i^{(0)}x_j^{(0)}, k=1,...,m \ , \qquad (2)$$

where $P_{ij,k}$ are the coefficients of heredity and

$$P_{ij,k}\geq 0,\ \sum_{k=1}^{m}P_{ij,k}=1,\ \ i,j,k=1,2,...,m\ .$$

Note that each element $x\in S^{m-1}$ is a probability distribution on $I=\{1,...,m\}$.

The association $x^{(0)}\to x^{(1)}$ means that the population evolves from an arbitrary state of probability distribution $x^{(0)}$, then passing to the state $x^{(1)}=V\left(x^{(0)}\right)$ which indicates the probability of the first generation,

the second generation $x^{(2)}=V\left(x^{(1)}\right)=V\left(V\left(x^{(0)}\right)\right)=V^2\left(x^{(0)}\right)$, and so on. Therefore, the evolution states of the population system can be described by the following discrete dynamical system,

$$x^{(0)},x^{(1)}=V\left(x^{(0)}\right),x^{(2)}=V^2\left(x^{(0)}\right),x^{(3)}=V^3\left(x^{(0)}\right),... \qquad (3)$$

where $V^n(x)=\underbrace{V(V(...V(x))...)}_{n}$ denotes the n times iteration of V to x.

Given that this operator involves the evolution of a free population from one generation to the next generation, thus it is also known as evolutionary operator. For a given $x^{(0)}\in S^{m-1}$, the trajectory $\left\{x^{(n)}\right\}$, $n=0,1,2,...$ of $x^{(0)}\in S^{m-1}$ under action of the mapping V in (2) is defined by $x^{(n+1)}=V\left(x^{(n)}\right)$, where $n=0,1,2,...$.

In other words, a distribution of the next generation can be described by quadratic stochastic operator if the distribution of the current generation is given. We should emphasize that the mapping V is a nonlinear (quadratic) operator, and it is higher-dimensional if $m\geq 3$. Higher dimensional dynamical systems are important but there are relatively few dynamical phenomena that are currently understood, for example, pendulum and solar system in mechanics, and evolution in biology.

The main problem for a given dynamical system (3) is to describe the limit points of $\left\{x^{(n)}\right\}_{n=0}^{\infty}$ for arbitrary given $x^{(0)}$.

Nonlinear operator theory falls within the general area of nonlinear functional analysis, an area which has become of interest in recent years. One of the central problems in the nonlinear operator theory is the asymptotical behavior of nonlinear operators. To study this problem, several classes of quadratic stochastic operator were constructed and investigated many publications. Similarly, limit behavior of the trajectories and the fixed points of the quadratic stochastic operator have been studied extensively by previous researchers, see (Akin & Losert, 1984; Akin, 1993; Bernstein, 1924; Ganikhodjaev *et al.*, 2015; Ganikhodjaev, 1993; Ganikhodjaev,1994; Ganikhodzhaev & Zanin, 2004; Ganikhodzhaev *et al.*, 2011; Hofbauer & Sigmund, 1998; Jenks, 1969; Kesten, 1970; Losert & Akin, 1983; Lyubich, 1978; Lyubich, 1992; Mukhamedov & Embong, 2015; Rozikov & Zhamilov, 2008; Ulam, 1960; Volterra, 1931; Zakharevich, 1978).

Another problem in nonlinear operator theory is the study of the asymptotical behavior of the trajectories. This problem was fully solved for Volterra quadratic stochastic operator, see (Ganikhodjaev, 1993; Ganikhodjaev, 1994). Furthermore, there are many publications devoted to the study of Volterra quadratic stochastic operator on finite and infinite state space, see (Akin & Losert, 1984; Ganikhodjaev & Hamzah, 2015b; Ganikhodzhaev *et al.*, 2011; Kesten, 1970; Lyubich, 1978; Lyubich, 1992; Mukhamedov *et al.*, 2005; Mukhamedov, 2000; Ulam, 1960; Zakharevich, 1978).

For the non-Volterra case, some new classes of quadratic stochastic operator have been constructed. In Ganikhodjaev & Hamzah (2014a, 2014b, 2015a, 2016), the authors introduced and studied Poisson, Gaussian, and Geometric quadratic stochastic operator on infinite state space. Ganikhodjaev & Hamzah (2015c) have introduced a new class of quadratic stochastic operator on the segment $[0,1]$ generated by 2-partition ξ, and it was shown that such operator is a regular transformation. Rozikov and Zhamilov (2008), and Mukhamedov and Embong (2015) also defined and introduced another class of quadratic stochastic operator. Then, the authors thoroughly described the properties as well as their trajectory behavior.

In the next section, the concept of quadratic stochastic operator on the set of all probability measures as well as the definition of Geometric quadratic stochastic opera or are presented in details.

A GEOMETRIC QUADRATIC STOCHASTIC OPERATOR

Let us recall some preliminaries.

Let (X,F) be a measurable space, and $S(X,F)$ be the set of all probability measures on (X,F), where X is a state space and F is σ - algebra on X. It is known that the set $S(X,F)$ is a compact, convex space and a form of Dirac measure δ_X which is defined by:

$$\delta_x(A)=\begin{cases}1 & \text{if } x\in A\\ 0 & \text{if } x\notin A\end{cases} \tag{4}$$

for any $A\in F$ are extremal elements of $S(X,F)$.

Definition 1: Suppose $(F,\leq)$ is a partially ordered set, M and m are elements of F, and A is a subset of F.

(i) M is a maximal element of A, if and only if M is in F and there is no x in A such that $M<x$; M is a maximum of A, if and only if M is in A and $x\leq M$ for all x in A.

(ii) m is a minimal element of A, if and only if m is in A and there is no x in A such that $x<m$; m is a minimum of A, if and only if m is in A and $m\leq x$ for all x in A.

Let $\{P(x,y,A): x,y\in X, A\in F\}$ be a family of functions on $X\times X\times F$ which satisfy the following conditions:

(i) $P(x,y,\cdot)\in S(X,F)$ for any fixed $x,y\in X$, that is, $P(x,y,\cdot): F\to[0,1]$ is the probability measure on F,

(ii) $P(x,y,A)$ is measurable function on $(X\times X, F\otimes F)$ which regarded as a function of two variables x and y with fixed $A\in F$,

(iii) $P(x,y,A)=P(y,x,A)$ for any $x,y\in X$ and $A\in F$.

Definition 2: A mapping $V: S(X,F)\to S(X,F)$ is called a quadratic stochastic operator generated by the family of functions $\{P(x,y,A): x,y\in X, A\in F\}$ if for an arbitrary measure $\mu\in S(X,F)$, then the measure $\mu'=V\mu$ is defined as follows:

$$\mu'(A)=\int_X\int_X P(x,y,A)\,d\mu(x)\,d\mu(y), \tag{5}$$

where $A\in F$ is an arbitrary measurable set.

Assume $\{V^n(\mu)\in S(X,F): n=0,1,2,...\}$ is a trajectory of the initial measure $\mu\in S(X,F)$, where $V^{n+1}(\mu)=V(V^n(\mu))$ for all $n=0,1,2,\ldots$, with $V^0(\mu)=\mu$.

Definition 3: A measure $\mu\in S(X,F)$ is called a fixed point of a quadratic stochastic operator V, if $V(\mu)=\mu$.

Definition 4: A quadratic stochastic operator V is called a *regular* if for any initial point $\mu\in S(X,F)$ the limit

$$\lim_{n\to\infty} V^n(\mu), \tag{6}$$

exists.

Note that the limit point is a fixed point of a quadratic stochastic operator V. Hence, the fixed points of quadratic stochastic operator manifest a limit behavior of the trajectories at any initial point.

If X is a state space, where $X=\{0,1,2,...,m\}$, thus for any $i,j\in X$, a probability measure $P(i,j,\cdot)$ is a discrete measure with $\sum_{i,j=1}^{m} P(ij,\{k\})=1$, where $P(ij,\{k\})=P_{ij,k}$ and corresponding stochastic operator V is defined as follows.

Definition 5: A mapping $V: S^{m-1}\to S^{m-1}$ is called a *quadratic stochastic operator*, if for any $x=(x_1,...,x_m)\in S^{m-1}$, Vx is defined as

$$(Vx)_k=\sum_{i,j=1}^{m} P_{ij,k}x_ix_j, \tag{7}$$

where the coefficients $P_{ij,k}$ satisfy the following conditions:

(i) $P_{ij,k}\geq 0$,

(ii) $P_{ij,k}=P_{ji,k}$,

(iii) $\sum_{k=1}^{m} P_{ij,k}=1$ for all $i,j,k\in\{1,...,m\}$.

It implies that these three conditions are fully consistent with conditions formulated in general case.

Note that, in this paper, we consider a countable state space X, where $X=\{0,1,2,...\}$. Hence, a quadratic stochastic operator V on measurable (X,F) is defined as follows:

$$V\mu(k)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}P_{ij,k}\mu(i)\mu(j), \tag{8}$$

where $k\in X$ for arbitrary measure $\mu\in S(X,F)$.

Geometric quadratic stochastic operator is considered in this paper. Note that a Geometric distribution G_r with a real parameter r, $0<r<1$, defined on X by the equation

$$G_r(k)=(1-r)r^k, \tag{9}$$

for any $k\in X$.

Let $S(X,F)$ be a set of all probability measure and $P(i,j,\cdot)$ be a probability measure on (X,F) for any $i,j\in X$.

Definition 6: A quadratic stochastic operator V in (7) is called a Geometric quadratic stochastic operator, if for any $i,j\in X$, the probability measure $P(i,j,\cdot)$ is the Geometric distribution $G_{r(i,j)}$ with a real parameter $r(i,j)=r(j,i),\ 0<r(i,j)<1$.

In this paper, we are motivated to construct a class of Geometric quadratic stochastic operator defined on countable state space X generated by 2-partition ξ of arbitrary singleton. The trajectory behavior of such operator with initial measure $\mu\in S(X,F)$ is studied and described.

A GEOMETRIC QUADRATIC STOCHASTIC OPERATOR GENERATED BY 2-PARTITION ξ OF SINGLETON

Let (X,F) be a measurable space with countable state space X.

Definition 7: A probabilistic measure μ on (X,F) is said to be discrete, there exists a finitely many elements $\{x_1,...,x_n\}\subset X$, such that $\mu(\{x_i\})=p_i$ for $i=1,...,n$, with $\sum_{i=1}^{n}p_i=1$. Then, $\mu(X\setminus\{\{x_1,...,x_n\}\})=0$ and for any $A\in F$, $\mu(A)=\sum_{x_i\in A}\mu(x_i)$.

Recall that a partition of (X,F), is a disjoint collection of elements of F whose union is X. We shall be interested in finite partitions. They will be denoted as $\xi=\{A_1,...,A_k\}$ and is called measurable k-partition.

Let $\xi=\{A_1,A_2\}$ be a measurable 2-partition of the state space $X=\{0,1,2,...\}$ where $A_1\subset X$, $A_2=X\setminus A_1$, and $\varsigma=\{B_1,B_2\}$ be a corresponding partition of the unit square $X\times X$, where $B_1=A_1\times A_1\cup A_2\times A_2$, and $B_2=A_1\times A_2\cup A_2\times A_1$. Note that the consideration of this partition is designed by the condition $P(i,j,\cdot)=P(j,i,\cdot)$.

We define the family $\{P(i,j,\cdot):i,j\in X\}$ of discrete probability measures on (X,F) as follows. If $(i,j)\in B_m$ where $m=1,2$, then

(i) for $(i,j)\in B_1$ assume $P(i,j,k)=(1-r_1)r_1^k$ for $k=0,1,2,...$

(ii) for $(i,j)\in B_2$ assume $P(i,j,k)=(1-r_2)r_2^k$ for $k=0,1,2,...$

Using the fact that arbitrary singleton in the countable state space X will result on different number of partitions in the unit square $X\times X$, thus we need to consider the following cases.

Case 1. Let $A_1=\{x_1:x_1\in X\}$ where A_1 consists of a singleton $x_1=0$ and $A_2=X\setminus A_1$. We consider a Geometric quadratic stochastic operator that

$$P_{ij,k}=\begin{cases}(1-r_1)r_1^k & \text{if } (i,j)\in B_1\\ (1-r_2)r_2^k & \text{if } (i,j)\in B_2\end{cases}$$

for $i,j\in X$.

Then, for any initial measure $\mu\in S(X,F)$, we have

$$\begin{aligned}
V\mu(k)&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}P_{ij,k}\mu(i)\mu(j)\\
&=P_{00,k}\mu(0)\mu(0)+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}P_{ij,k}\mu(i)\mu(j)+\sum_{i=1}^{\infty}P_{i0,k}\mu(i)\mu(0)\\
&+\sum_{j=1}^{\infty}P_{0j,k}\mu(0)\mu(j)\\
&=(1-r_1)r_1^k\left[\mu(0)\right]^2+(1-r_1)r_1^k\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mu(i)\mu(j)\\
&+(1-r_2)r_2^k\mu(0)\sum_{i=1}^{\infty}\mu(i)+(1-r_2)r_2^k\mu(0)\sum_{j=1}^{\infty}\mu(j)\\
&=(1-r_1)r_1^k\left[\mu(0)\right]^2+(1-r_1)r_1^k\left[1-\mu(0)\right]^2\\
&+2(1-r_2)r_2^k\mu(0)\left[1-\mu(0)\right]\\
&=(1-r_1)r_1^k\left\{\left[\mu(0)\right]^2+\left[1-\mu(0)\right]^2\right\}\\
&+(1-r_2)r_2^k\left\{2\mu(0)\left[1-\mu(0)\right]\right\}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
V^2\mu(k)&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}P_{ij,k}V\mu(i)V\mu(j)\\
&=P_{00,k}V\mu(0)V\mu(0)+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}P_{ij,k}V\mu(i)V\mu(j)+\sum_{i=1}^{\infty}P_{i0,k}V\mu(i)V\mu(0)\\
&+\sum_{j=1}^{\infty}P_{0j,k}V\mu(0)V\mu(j)\\
&=(1-r_1)r_1^k\left[V\mu(0)\right]^2+(1-r_1)r_1^k\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}V\mu(i)V\mu(j)\\
&+(1-r_2)r_2^kV\mu(0)\sum_{i=1}^{\infty}V\mu(i)+(1-r_2)r_2^kV\mu(0)\sum_{j=1}^{\infty}V\mu(j)\\
&=(1-r_1)r_1^k\left[V\mu(0)\right]^2+(1-r_1)r_1^k\left[1-V\mu(0)\right]^2\\
&+2(1-r_2)r_2^kV\mu(0)\left[1-V\mu(0)\right]\\
&=(1-r_1)r_1^k\left[\left(V\mu(0)\right)^2+\left(1-V\mu(0)\right)^2\right]\\
&+(1-r_2)r_2^k\left[2V\mu(0)\left(1-V\mu(0)\right)\right].
\end{aligned}$$

Thus, by using induction on the sequence $V^n\mu(k)$, we produce the following recurrent equation:

$$\begin{aligned}
&V^{n+1}\mu(k)\\
&=(1-r_1)r_1^k\left[\left(V^n\mu(0)\right)^2+\left(1-V^n\mu(0)\right)^2\right]\\
&+(1-r_2)r_2^k\left[2V^n\mu(0)\left(1-V^n\mu(0)\right)\right],
\end{aligned} \tag{10}$$

where $n=0,1,2,...$.

One can show that the limit behavior of the recurrent equation (9) is fully determined by limit behavior of recurrent equation $V^n\mu(0)$ such that

$$V^{n+1}\mu(0)$$
$$=(1-r_1)\left[\left(V^n\mu(0)\right)^2+\left(1-V^n\mu(0)\right)^2\right]$$
$$+(1-r_2)\left[2V^n\mu(0)\left(1-V^n\mu(0)\right)\right], \tag{11}$$

for $n=0,1,2,\ldots$.

Case 2. Let $A_1=\{x_1 : x_1\in X\}$ where A_1 consists of a singleton $x_1\neq 0$ and $A_2=X\setminus A_1$. Thus, for any initial measure $\mu\in S(X,F)$, we have

$$V\mu(k)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}P_{ij,k}\mu(i)\mu(j)$$
$$=P_{x_1x_1,k}\mu(x_1)\mu(x_1)+\sum_{i=0}^{x_1-1}\sum_{j=0}^{x_1-1}P_{ij,k}\mu(i)\mu(j)+\sum_{i=x_1+1}^{\infty}\sum_{j=x_1+1}^{\infty}P_{ij,k}\mu(i)\mu(j)$$
$$+\sum_{i=0}^{x_1-1}\sum_{j=x_1+1}^{\infty}P_{ij,k}\mu(i)\mu(j)+\sum_{i=x_1+1}^{\infty}\sum_{j=0}^{x_1-1}P_{ij,k}\mu(i)\mu(j)+\sum_{i=0}^{x_1-1}P_{ih,k}\mu(i)\mu(x_1)$$
$$+\sum_{j=0}^{x_1-1}P_{x_1j,k}\mu(x_1)\mu(j)+\sum_{j=x_1+1}^{\infty}P_{x_1j,k}\mu(x_1)\mu(j)+\sum_{i=x_1+1}^{\infty}P_{ix_1,k}\mu(i)\mu(x_1)$$
$$=(1-r_1)r_1^k\left[\mu(x_1)\right]^2+(1-r_1)r_1^k\left[\sum_{i=0}^{x_1-1}\mu(i)\right]\left[\sum_{j=0}^{x_1-1}\mu(j)\right]$$
$$+(1-r_1)r_1^k\left[1-\sum_{i=0}^{x_1}\mu(i)\right]\left[1-\sum_{j=0}^{x_1}\mu(j)\right]$$
$$+(1-r_1)r_1^k\left[\sum_{i=0}^{x_1-1}\mu(i)\right]\left[1-\sum_{j=0}^{x_1}\mu(j)\right]$$
$$+(1-r_1)r_1^k\left[1-\sum_{i=0}^{x_1}\mu(i)\right]\left[\sum_{j=0}^{x_1-1}\mu(j)\right]+(1-r_2)r_2^k\mu(x_1)\sum_{i=0}^{x_1-1}\mu(i)$$
$$+(1-r_2)r_2^k\mu(x_1)\sum_{j=0}^{x_1-1}\mu(i)+(1-r_2)r_2^k\mu(x_1)\left[1-\sum_{j=0}^{x_1}\mu(j)\right]$$
$$+(1-r_2)r_2^k\mu(x_1)\left[1-\sum_{i=0}^{x_1}\mu(i)\right]$$
$$=(1-r_1)r_1^k\left[\left(\mu(x_1)\right)^2+\left(1-\mu(x_1)\right)^2\right]$$
$$+(1-r_2)r_2^k\left[2\mu(x_1)\left(1-\mu(x_1)\right)\right], \text{ and}$$

$$V^2\mu(k)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}P_{ij,k}V\mu(i)V\mu(j)$$
$$=P_{x_1x_1,k}V\mu(x_1)V\mu(x_1)+\sum_{i=0}^{x_1-1}\sum_{j=0}^{x_1-1}P_{ij,k}V\mu(i)V\mu(j)$$
$$+\sum_{i=x_1+1}^{\infty}\sum_{j=x_1+1}^{\infty}P_{ij,k}V\mu(i)V\mu(j)+\sum_{i=0}^{x_1-1}\sum_{j=x_1+1}^{\infty}P_{ij,k}V\mu(i)V\mu(j)$$
$$+\sum_{i=x_1+1}^{\infty}\sum_{j=0}^{x_1-1}P_{ij,k}V\mu(i)V\mu(j)+\sum_{i=0}^{x_1-1}P_{ix_1,k}V\mu(i)V\mu(x_1)$$
$$+\sum_{j=0}^{x_1-1}P_{x_1j,k}V\mu(x_1)V\mu(j)+\sum_{j=x_1+1}^{\infty}P_{x_1j,k}V\mu(x_1)V\mu(j)$$
$$+\sum_{i=x_1+1}^{\infty}P_{ix_1,k}V\mu(i)V\mu(x_1)$$
$$=(1-r_1)r_1^k\left[V\mu(x_1)\right]^2+(1-r_1)r_1^k\left[\sum_{i=0}^{x_1-1}V\mu(i)\right]\left[\sum_{j=0}^{x_1-1}V\mu(j)\right]$$
$$+(1-r_1)r_1^k\left[1-\sum_{i=0}^{x_1}V\mu(i)\right]\left[1-\sum_{j=0}^{x_1}V\mu(j)\right]$$
$$+(1-r_1)r_1^k\left[\sum_{i=0}^{x_1-1}V\mu(i)\right]\left[1-\sum_{j=0}^{x_1}V\mu(j)\right]$$
$$+(1-r_1)r_1^k\left[1-\sum_{i=0}^{x_1}V\mu(i)\right]\left[\sum_{j=0}^{x_1-1}V\mu(j)\right]+(1-r_2)r_2^kV\mu(x_1)\sum_{i=0}^{x_1-1}V\mu(i)$$
$$+(1-r_2)r_2^kV\mu(x_1)\sum_{j=0}^{x_1-1}V\mu(i)+(1-r_2)r_2^kV\mu(x_1)\left[1-\sum_{j=0}^{x_1}V\mu(j)\right]$$
$$+(1-r_2)r_2^kV\mu(x_1)\left[1-\sum_{i=0}^{x_1}V\mu(i)\right]$$
$$=(1-r_1)r_1^k\left[\left(V\mu(x_1)\right)^2+\left(1-V\mu(x_1)\right)^2\right]$$
$$+(1-r_2)r_2^k\left[2V\mu(x_1)\left(1-V\mu(x_1)\right)\right].$$

By using induction on the sequence $V^n\mu(k)$, the following recurrent equation is obtained

$$V^{n+1}\mu(k)$$
$$=(1-r_1)r_1^k\left[\left(V^n\mu(x_1)\right)^2+\left(1-V^n\mu(x_1)\right)^2\right]$$
$$+(1-r_2)r_2^k\left[2V^n\mu(x_1)\left(1-V^n\mu(x_1)\right)\right], \tag{12}$$

where $n=0,1,2,\ldots$.

From this, it is clear that the limit behavior of the recurrent equation (12) is fully determined by limit behavior of recurrent equation $V^n\mu(x_1)$ such that

$$V^{n+1}\mu(x_1)$$
$$=(1-r_1)r_1^{x_1}\left[\left(V^n\mu(x_1)\right)^2+\left(1-V^n\mu(x_1)\right)^2\right]$$
$$+(1-r_2)r_2^{x_1}\left[2V^n\mu(x_1)\left(1-V^n\mu(x_1)\right)\right], \tag{13}$$

for $n=0,1,2,\ldots$.

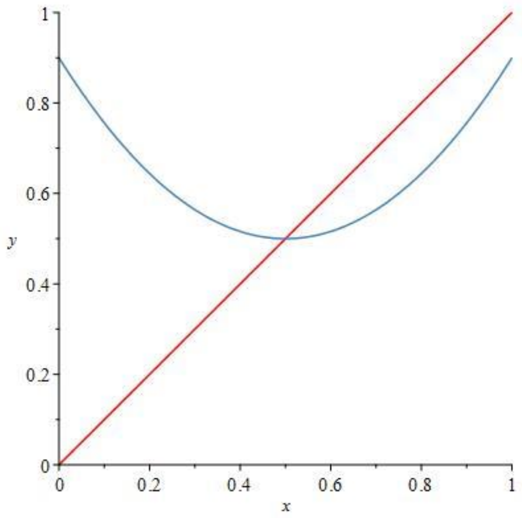
In other words, for both cases, we can generalize that for any singleton $x_1\in A_1$ the recurrent equation in (12) is fully determined by limit behavior of recurrent equation in (13).

As $n\to\infty$, then the recurrent equation (13) can be written as follows:

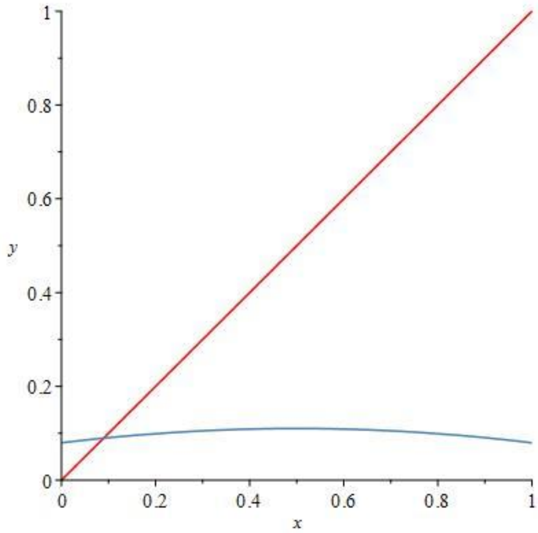
$$x=(1-r_1)r_1^{x_1}\left[x^2+(1-x)^2\right]+(1-r_2)r_2^{x_1}\left[2x(1-x)\right], \tag{14}$$

where $x_1\in A_1$.

By solving the equation in (14), we have single fixed point x_1^* as in Fig. 1.



(a) Graph 1.1 when $r_1 = 0.1$ and $r_2 = 0.9$ for $x_1 = 0$



(b) Graph 1.2 when $r_1 = 0.35$ and $r_2 = 0.75$ for $x_1 = 2$

Fig. 1 Graph of the Function in (13) for Some Fixed Values r_1 and r_2.

From Fig. 1, using simple calculus, one can show that the trajectory behavior of quadratic stochastic operator in (12) which defined on zero-dimensional simplex, S^0 converges to this fixed point x_1^*. Then, we can say that the quadratic stochastic operator in (12) is regular. Thus, for any initial measure μ, we have:

$$\lim_{n\to\infty} V^n \mu(x_1) = x_1^* .$$

Then, passing the limit in (12), for any singleton k, we have

$$\lim_{n\to\infty} V^{n+1}\mu(k)$$

$$= \lim_{n\to\infty} \begin{cases} (1-r_1)r_1^k\left[\left(V^n\mu(x_1)\right)^2 + \left(1-V^n\mu(x_1)\right)^2\right] \\ +(1-r_2)r_2^k\left[2V^n\mu(x_1)\left(1-V^n\mu(x_1)\right)\right] \end{cases}$$

$$= (1-r_1)r_1^k \cdot \left[\left(x_1^*\right)^2 + \left(1-x_1^*\right)^2\right] + (1-r_2)r_2^k \cdot \left[2x_1^*\left(1-x_1^*\right)\right]$$

$$= G_{r_1}(k)\cdot\left[\left(x_1^*\right)^2 + \left(1-x_1^*\right)^2\right] + G_{r_2}(k)\cdot\left[2x_1^*\left(1-x_1^*\right)\right].$$

Thus, for any initial measure μ, the strong limit of the sequence $V^n\mu$ exists and is equal to the convex linear combination

$$\lim_{n\to\infty} V^{n+1}\mu(k)$$

$$= \left[\left(x_1^*\right)^2 + \left(1-x_1^*\right)^2\right] G_{r_1}(k) + \left[2x_1^*\left(1-x_1^*\right)\right] G_{r_2}(k),$$

of two Geometric measures G_{r_1} and G_{r_2}. It is clear that

$$Fix(V) = \left[\left(x_1^*\right)^2 + \left(1-x_1^*\right)^2\right] G_{r_1}(k) + 2x_1^*\left(1-x_1^*\right) G_{r_2}(k) .$$

By Definition 4, the convergence of the trajectory indicates that the limit exists. Hence, it is regular.

In the next section, we provide the analytical proof of Geometric quadratic stochastic operator generated by 2-partition ξ of a singleton.

Analytical proof of a geometric quadratic stochastic operator generated by 2-partition ξ of singleton

Based on the equation (14), it is obvious that the equation is a quadratic equation. Hence, we can rewrite the right-hand side equation as follows:

$$y = 2\left[(1-r_1)r_1^{x_1} - (1-r_2)r_2^{x_1}\right]x^2 - 2\left[(1-r_1)r_1^{x_1} - (1-r_2)r_2^{x_1}\right]x + (1-r_1)r_1^{x_1}, \quad (15)$$

where $x_1 \in A_1$, and $0 < r_1, r_2 < 1$. It is obvious that a function (15) maps the segment $[0,1]$ (one-dimensional simplex) into itself with

$$y|_{x=0} = y|_{x=1} = (1-r_1)r_1^{x_1} .$$

Since we have assumed that $0 < r_1 < 1$ to avoid the analysis of particularities, then the following statements are valid.

Theorem 1: A fixed point of the transformation (15) is a unique and belongs to open interval $(0,1)$.

Proof In fact, the equation

$$x = 2(A-B)x^2 - 2(A-B)x + A, \quad \left(A = (1-r_1)r_1^{x_1}; B = (1-r_2)r_2^{x_1}\right), \quad (16)$$

has a root in the interval $(1,\infty)$ when $2(A-B) > 0$, and has a root in the interval $(-\infty,0)$ when $2(A-B) < 0$. If $2(A-B) = 0$, then we can clearly see that the equation became a linear with $A = (1-r_1)r_1^{x_1} > 0$. Thus, for all cases, a root in $[0,1]$ is unique. It is evident that this root differs from 0 to 1.

Now, let us consider the discriminant of the quadratic equation (16) to investigate the local character of the fixed point, where

$$\Delta = 4(1-A)A + (1-2B)^2 . \quad (17)$$

By using simple calculus, we have that $0 < \Delta < 2$, and Δ takes all value in this interval.

Theorem 2: If $0 < \Delta < 2$, then a fixed point is attractive.

Proof Let ζ is a fixed point, where

$$\zeta = \frac{2(A-B) + 1 - \sqrt{\Delta}}{4(A-B)} . \quad (18)$$

Its character is defined by $f'(\zeta)$, where $f(x)$ is a right hand side of the equation (16), and $f'(x)$ is its derivative. Let $\lambda = f'(\zeta)$, where

$$\lambda = 4(A-B)\zeta - 2(A-B) \ . \tag{19}$$

It is easy to verify that

$$\lambda = 1-\sqrt{\Delta} \ , \tag{20}$$

for a fixed point in the interval $(0,1)$. Since $0<\Delta<2$, then we will have $1-\sqrt{2}<\lambda<1$. Note that, if $|\lambda|<1$, then ζ is an attractive point, and if $|\lambda|>1$, then ζ is a repelling point. Thus, any unique fixed point in the open interval $(0,1)$ is attractive, and the statement of the Theorem 2 follows from the equality in (20).

It is shown that the trajectory behavior of quadratic stochastic operator in (13) converges to a fixed point in the open interval $(0,1)$.

Proposition: A Geometric quadratic stochastic operator generated by 2-partition ξ of a singleton is a regular transformation.

CONCLUSION

A limit behavior of Geometric quadratic stochastic operator generated by 2-partition ξ of arbitrary singleton is a regular transformation.

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